

Taylor Series

1. Determine the fourth-order Taylor polynomial of $f(x) = \cos(x)$ centered at 0.
2. Consider the function $f(x) = \sin x$. Approximate $f(\pi/8)$ and $f(10\pi/9)$ using the fifth-order Taylor polynomial centered at 0.
3. What is the second term in the Taylor series of $\sqrt[4]{4x-1}$ about 4.25?
4. In the Taylor series (about $\pi/4$) for the function $\sin x + \cos x$, find the third nonzero term.
5. In the Taylor series for the function $3x^2 - 7 + \cos x$ (expanded in power of x), what is the coefficient of x^2 ?

Numerical Differentiation

1. Calculate the first-order derivative of the function $g(x) = e^{\sin(x)}$ at the point $x = 0.5$ using first-order finite differences (forward, backward, and central) with $h = 0.01$.
2. Calculate the second-order derivative of the function $g(x) = e^{\cos(x)}$ at the point $x = 0.5$ using second-order finite differences (forward, backward, and central) with $h = 0.01$.
3. The distance traveled in meters by a rocket at each second is given by the following values:

t [s]	0	1	2	3	4	5
y [m]	0.0	2.5	7.8	18.2	51.9	80.3

Use numerical differentiation to approximate the velocity and acceleration at each point.

4. Determine the second-order Taylor polynomial of $f(x) = x^x \cos(e^{x^2}) + \ln(\sin(x^3))$ centered at 1. Use numerical differentiation instead of analytical differentiation.
5. Consider the function $f(x) = \sin(x)$ at $x_0 = 1.0$, $x_1 = 1.4$, and $x_2 = 2.0$. Approximate the derivative $f'(x_1)$ at x_1 using the divided difference formula. Compare this approximation with the exact derivative.

Zeros

1. Using the bisection method, determine the point of intersection of the curves given by

$$y = x^3 - 2x + 1 \quad \text{and} \quad y = x^2.$$

2. Find a root of the equation $6(e^x - x) = 6 + 3x^2 + 2x^3$ between -1 and $+1$ using the bisection method.
3. Find a root of the equation $6(e^x - x) = 6 + 3x^2 + 2x^3$ between -1 and $+1$ using the false position method.
4. Use the false position method to find a zero of the equation

$$\lambda \cosh\left(\frac{50}{\lambda}\right) = \lambda + 10$$

5. If Newton's method is used on $f(x) = x^3 - x + 1$ starting with $x_0 = 1$, what will x_1 be?
6. Using the Newton's method, locate the root of $f(x) = e^{-x} - \cos x$ that is nearest $\pi/2$.
7. Carry out three iterations of Newton's method using $x_0 = 1$ and $f(x) = 3x^3 + x^2 - 15x + 3$.
8. If the secant method is used on $f(x) = x^5 + x^3 + 3$ and if $x_{n-2} = 0$ and $x_{n-1} = 1$, what is x_n ?
9. Use the secant method to find the zero near -0.5 of $f(x) = e^x - 3x^2$.
10. Knowing that $f(x) = \frac{\cos(\ln x)}{e^x}$, use the secant method to find the zero near 0.25 .

Extrema of Functions

1. Use the golden section algorithm to determine the minimum of $f(x) = 2x^3 - 9x^2 + 12x + 2$ on $[0, 3]$.
2. Consider the function $f(x) = -1.5x^6 - 2.4x^4 + 12x$. Estimate the value of $x \in [0, 2]$ that maximizes $f(x)$ by applying the quadratic interpolation method three times with $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$.
3. Find an estimate for the minimum of $f(x) = x + \frac{1}{x}$ using the quadratic interpolation method with $x_0 = 0.1$, $x_1 = 0.5$, $x_2 = 1.0$.
4. Find an estimate of the maximizer of $f(x) = 2x^3 - 9x^2 + 12x$ in $[0, 2]$ using Newton's method with $x_0 = 2$ and iterating 3 times.
5. Using the Newton's method, locate the minimum of $f(x) = 0.01x^2 + \frac{500000}{0.25x^2}$ starting with $x_0 = 10$.

Systems of Equations

1. Solve the system of equations

$$\begin{cases} 3x_1 + 4x_2 + 3x_3 = 10 \\ x_1 + 5x_2 - x_3 = 7 \\ 6x_1 + 3x_2 + 7x_3 = 15 \end{cases},$$

using naive Gaussian elimination, Gaussian elimination with partial and complete pivoting.

2. Solve the system of equations

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix},$$

using the Gauss-Jordan elimination.

3. Calculate the ℓ_1 , ℓ_2 and ℓ_∞ norms of

$$[10 \ 7 \ 15]^T$$

4. Calculate the ℓ_1 , ℓ_F and ℓ_∞ norms of

$$\begin{bmatrix} 3 & 4 & 3 \\ 1 & 5 & -1 \\ 6 & 3 & 7 \end{bmatrix}$$

5. Let

$$\begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 4 \\ 1 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ 11 \end{bmatrix}.$$

Carry out a number of iterations of the Jacobi iteration, starting with the $[1, 1, 1]^T$ initial vector. Start by verifying diagonal dominance.

6. Let

$$\begin{bmatrix} -2 & 1 & 0 \\ -1 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ -5 \end{bmatrix}.$$

Carry out a number of iterations of the Gauss-Seidel iteration, starting with the zero initial vector. Start by verifying diagonal dominance.

7. Use the Newton's method to solve the system of equations

$$\begin{cases} x + y + z = 3 \\ x^2 + y^2 + z^2 = 5 \\ e^x + xy - xz = 1 \end{cases},$$

starting with $[0.1, 1.2, 2.5]^T$.

8. Solve this pair of simultaneous nonlinear equations by first eliminating y and then solving the resulting equation in x by Newton's method. Start with the initial value $x_0 = 1.0$.

$$\begin{cases} x^3 - 2xy + y^7 - 4x^3y = 5 \\ y \sin x + 3x^2y + \tan x = 4 \end{cases}$$

Interpolation & Approximation

1. Use the Lagrange interpolation process to obtain a polynomial of least degree that assumes these values:

x	0	2	3	4
y	7	11	28	63

2. Use the Newton interpolation process to obtain a polynomial of least degree that assumes these values:

x	1	3	-2	4	5
y	2	6	-1	-4	2

3. Determine the natural cubic spline that interpolates the function at the given points.

x	0	0.5	0.7	1.0
y	0	0.6	1.2	2.2

4. Using the least-square method, fit the data in the table

x	0.50	0.55	0.60	0.65	0.70	0.75	0.80
y	1.2	1.0	0.7	0.4	0.1	-0.2	-0.6

by a parabola.

5. Fit a function of the form $y = a \ln x + b \cos x + ce^x$ to the following table values

x	0.24	0.65	0.95	1.24	1.73	2.01	2.23	2.52	2.77	2.99
y	0.23	-0.26	-1.10	-0.45	0.27	0.10	-0.29	0.24	0.56	1.00

6. Using the least-square method, fit the data in the table

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
y	0.6	1.1	1.6	1.8	2.0	1.9	1.7	1.3

by a function $y = a \sin(bx)$. Do it with and without linearization.

Numerical Integration

1. What is the numerical value of the composite trapezoid rule applied to the reciprocal function $f(x) = x^{-1}$ using the points 1, 4/3, 2?
2. Approximate $\int_0^2 2^x \, dx$ using the composite trapezoid rule with $h = 1/2$.
3. Approximate $\int_1^2 f(x) \, dx$ given the table of values

x	1	5/4	3/2	7/4	2
$f(x)$	10	8	7	6	5

Compute an estimate by the composite trapezoid rule.

4. Use the composite trapezoidal and Simpson's rules with 4 sub-intervals to approximate the following integral and determine an upper bound for the error.

$$\int_1^2 x \ln(x) \, dx$$

5. Consider the integral

$$\int_1^{1.6} \frac{2x}{x^2 - 4} \, dx$$

Apply all of the Newton-Cotes closed rules.

6. Consider the integral

$$\int_{-1}^1 \frac{1}{\sqrt{1 - x^2}} \, dx.$$

Because it has singularities at the endpoints of the interval $[-1, 1]$, closed rules cannot be used. Apply all of the Newton-Cotes open rules. Compare and explain these numerical results to the true solution, which is

$$\int_{-1}^1 (1 - x^2)^{-1/2} \, dx = \arcsin x \Big|_{-1}^1 = \pi.$$

Ordinary Differential Equations

1. Consider the initial value problem

$$\begin{cases} y' = -ty \\ y(0) = 1 \end{cases}, \quad t \in [0, +\infty[. \quad (\text{solution : } y(t) = e^{-t^2/2}).$$

Calculate an approximation for the solution at $t = 0.5$, using the Euler method with step $h = 0.1$.

2. Given the initial value problem

$$\begin{cases} y' = \frac{y}{1+t^2} \\ y(0) = 1 \end{cases}, \quad 0 \leq t \leq 1 \quad (\text{solution : } y(t) = e^{\arctan(t)}).$$

Use the second-order Taylor method to compute an approximate value y_5 for $y(1)$.

3. Using Euler's method, compute an approximate value for $x(2)$ for the differential equation

$$x' = 1 + x^2 + t^3,$$

with the initial value $x(1) = -4$ using 100 steps.

4. Consider the ordinary differential equation

$$\begin{cases} x' = t^2 + tx' - 2xx' \\ x(0) = 1 \end{cases}.$$

Take three steps using the Runge-Kutta method of order 2 with step $h = 0.1$.

5. Given the initial value problem

$$\begin{cases} y' = e^t y^2 + e^3 \\ y(2) = 4 \end{cases}.$$

Use the fourth-order Runge-Kutta method to compute an approximate value y_5 for $y(5)$.

6. Find approximations for the solution of the initial value problem

$$\begin{cases} y' = -ty^2 \\ y(1) = 2 \end{cases},$$

for $t \in [1, 2]$, with $h = 0.2$. Use the 2th and 4th order Adams-Bashforth-Moulton predictor-corrector method.

Systems of Ordinary Differential Equations

1. Given the system

$$\begin{cases} x' = 2x - 3y \\ y' = y - 2x \\ x(0) = 8, \quad y(0) = 3 \end{cases},$$

apply Euler's method to estimate $x(0.1)$ and $y(0.1)$ with two integration steps.

2. Use the 2th order Runge-Kutta method, with $h = 0.05$, to estimate the values $y(0.1)$ and $x(0.1)$ of the particular solution of

$$\begin{cases} x' = 2t - 3y \\ y' = t - x^2 \end{cases},$$

with $x(0) = 2$ and $y(0) = 1$.

3. Convert the third-order differential equation

$$y''' + ty'' - yy' + y^2 = \cos(t)$$

to a system of first-order differential equations with initial conditions

$$y(1.2) = -1, \quad y'(1.2) = 0, \quad y''(1.2) = 1.$$

4. Determine an approximate value for $y(0.5)$ using the Euler and second-order Taylor methods for the initial value problem

$$y'' + 2y' + 5y = e^{-t} \sin(t), \quad y(0) = 0, \quad y'(0) = 1$$

5. Determine an approximate value for $y(0.5)$ using the second-order Adams-Basforth-Moulton method for the initial value problem

$$y'' - 3y' + 2y = 4e^{2t}, \quad y(0) = -3, \quad y'(0) = 5$$