Data Science in Aerospace

Jointly Distributed Random Variables

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1 Landing Performance

An aircraft performs three consecutive flights. Let X be the number of successful landings in the first two flights, and let Y be the number of unsuccessful landings in the last two flights. Assume that both events have equal probability.

- Define the joint probability function of the variables X and Y.

$$X: \text{ number of successful landings in the first two flights}$$

$$Y: \text{ number of unsuccessful landings in the last two flights}$$

$$P(X,Y)$$

$$\begin{array}{lll} P(0,0)=0 & P(1,0)=1/8 & P(2,0)=1/8 \\ P(0,1)=1/8 & P(1,1)=2/8 & P(2,1)=1/8 \\ P(0,2)=1/8 & P(1,2)=1/8 & P(2,2)=0 \end{array}$$

- Define the conditional probability function of Y, given that $X=1$.

$$P_{Y|X=1}(Y=0)=\frac{P(Y=0,X=1)}{P(X=1)}=\frac{1/8}{1/2}=1/4$$
$$P_{Y|X=1}(Y=1)=\frac{P(Y=1,X=1)}{P(X=1)}=\frac{2/8}{1/2}=1/2$$
$$P_{Y|X=1}(Y=2)=\frac{P(Y=2,X=1)}{P(X=1)}=\frac{1/8}{1/2}=1/4$$

- Calculate the correlation coefficient ρ_{XY}.

$$\text{Corr}(X,Y)=\frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

$$\mu_X = \sum_x x P_X(x) = 0 \times (0 + 1/8 + 1/8)$$
$$+ 1 \times (1/8 + 2/8 + 1/8)$$
$$+ 2 \times (1/8 + 1/8 + 0) = 1$$
$$\mu_Y = \sum_y y P_Y(y) = 1$$

$$\sigma_X^2 = \sum_x (x - \mu_X)^2 P_X(x) = 1/\sqrt{2}$$
$$\sigma_Y^2 = \sum_y (y - \mu_Y)^2 P_Y(y) = 1/\sqrt{2}$$

Test if X and Y are independent

$$P_{X,Y}(2,0) = P_X(2)P_Y(0) \Leftrightarrow 1/8 = 1/16 \Leftrightarrow \text{They are not independent}$$

$$\text{Cov}(X,Y) = \sum_x \sum_y (x - \mu_X)(y - \mu_Y) P_{XY}(X,Y) = -1/4$$

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{-1/4}{1/\sqrt{2}} \approx -0.35$$

2 Departures

An airport monitors the number of flights departing daily from two airlines, Airline X and Airline Y. The joint probability function of the number of flights departing daily is as follows:

X/Y	0	1	2
0	0.12	0.25	0.13
1	0.05	0.30	0.01
2	0.03	0.10	0.01

- Calculate the marginal probability functions of X and Y.
- Calculate the marginal distribution function of X.
- Calculate the probability that, on a given day, Airline Y has more departing flights than Airline X.
- Determine the expected value and variance of the total number of flights departing daily.

3 Flight Parameters

Let X and Y represent random variables related to aviation, where X denotes the wind speed (in knots) and Y denotes the altitude level (in thousands of feet). Their joint probability function is given by:

X/Y	-1	0	1
-1	0	$1/4$	0
0	$1/4$	0	$1/4$
1	0	$1/4$	0

- Show that Cov$(X, Y) = 0$ but that X and Y are not independent.

4 Pilot Certification

To qualify for a pilot certification, a candidate must complete two independent tests, A (theoretical test) and B (practical flight test). The performance in each test is classified as insufficient (0), sufficient (1), or excellent (2). The probability of the candidate achieving scores of 0, 1, or 2 in tests A and B is provided in the following table:

Score	Test A	Test B
0	0.2	0.2
1	0.5	0.6
2	0.3	0.2

Consider the random pair (X, Y), where X is the absolute difference between the scores obtained in tests A and B and Y is the sum of the scores obtained in tests A and B.

- Determine the joint probability function of the random pair (X, Y).

X : is the absolute difference between the scores obtained in tests A and B

Y : sum of the scores obtained in tests A and B

$$P(X, Y)$$

$$P(0,0) = 0.2 \times 0.2 = 0.04$$
$$P(0,1) = 0$$
$$P(0,2) = 0.5 \times 0.6 = 0.3$$
$$P(0,3) = 0$$
$$P(0,4) = 0.3 \times 0.2 = 0.06$$

$$P(1,0) = 0$$
$$P(1,1) = 0.2 \times 0.6 + 0.5 \times 0.2 = 0.22$$
$$P(1,2) = 0$$
$$P(1,3) = 0.5 \times 0.2 + 0.3 \times 0.6 = 0.28$$
$$P(1,4) = 0$$

$$P(2,0) = 0$$
$$P(2,1) = 0$$
$$P(2,2) = 0.2 \times 0.2 + 0.3 \times 0.2 = 0.1$$
$$P(2,3) = 0$$
$$P(2,4) = 0$$

X/Y	0	1	2	3	4	P_X
0	0.04	0	0.30	0	0.06	0.4
1	0	0.22	0	0.28	0	0.5
2	0	0	0.1	0	0	0.1
P_Y	0.04	0.22	0.4	0.28	0.06	

- Determine the marginal probability functions of X and Y.

$$P_X(X=0) = 0.4$$
$$P_X(X=1) = 0.5$$
$$P_X(X=3) = 0.1$$

$$P_Y(Y=0) = 0.04$$
$$P_Y(Y=1) = 0.22$$
$$P_Y(Y=2) = 0.4$$
$$P_Y(Y=3) = 0.28$$
$$P_Y(Y=4) = 0.06$$

- Determine the cumulative distribution function of X.

$$F_X(0) = 0.4$$
$$F_X(1) = 0.9$$
$$F_X(2) = 1.0$$

- Determine the conditional probability function of X given that $Y = 2$.

$$P_{X|Y=2}(X=0) = \frac{P(X=0, Y=2)}{P(Y=2)} = \frac{0.3}{0.4} = 0.75$$
$$P_{X|Y=2}(X=1) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{0}{0.4} = 0$$
$$P_{X|Y=2}(X=2) = \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{0.1}{0.4} = 0.25$$

- State, with justification, whether X and Y are independent.
  - Test if X and Y are independent

$$P_{XY}(0,0) = P_X(0)P_Y(0) \Leftrightarrow 0.04 = 0.016 \Leftrightarrow \text{They are not independent}$$

- Calculate all conditional probability functions of Y given X.

$$\begin{aligned}
P_{Y|X}(Y=1) =& \frac{P(Y=1, X=0)}{P_X(X=0)} \\
P_{Y|X}(Y=1) =& \frac{P(Y=1, X=1)}{P_X(X=1)} \\
P_{Y|X}(Y=1) =& \frac{P(Y=1, X=2)}{P_X(X=2)} \\
&\vdots \\
P_{Y|X}(Y=4) =& \frac{P(Y=4, X=2)}{P_X(X=2)}
\end{aligned}$$

- Calculate the expected value $E[Y|X=2]$ and variance Var$[Y|X=2]$.

$$E[Y|X=2] = \sum_y y P_Y(Y=y \mid X=2)$$

$$\text{Var}(Y|X=2) = \sum_y (y - \mu_{Y|X=2})^2 P_Y(Y=y \mid X=2)$$

- Calculate $F_{Y|X=0}(y)$.

$$\begin{aligned}
F_{Y|X=0}(0) &= 0.04 \\
F_{Y|X=0}(1) &= 0.04 \\
F_{Y|X=0}(2) &= 0.34 \\
F_{Y|X=0}(3) &= 0.34 \\
F_{Y|X=0}(4) &= 0.40
\end{aligned}$$

- Calculate the probability that $Y=2$ given that $X \cdot Y = 0$.

$$P = \frac{P(0,2)}{P(X=0 \vee Y=0)} = 0.75$$

- Calculate the probability that $X+Y$ is odd.

$$P = 0$$

5 Cruise Phase

Consider two random variables X and Y that represent the altitude (in tens of thousands of feet) and airspeed (in hundreds of knots) of an aircraft during cruise, respectively. The joint probability density function is given by:

$$f(x,y) = \begin{cases} c\,x\,y, & \text{if } (x,y) \in D, \\ 0, & \text{if } (x,y) \notin D, \end{cases}$$

where $D = \{(x,y) \in \mathbb{R}^2 : 3 \leq x \leq 4,\ 4 \leq y \leq 5\}$.

- Determine the value of the constant c such that $f(x,y)$ is a valid probability density function.

$$\iint_D c\,x\,y\,\mathrm{d}A = 1$$

$$\int_3^4 \int_4^5 cxy\,\mathrm{d}x\,\mathrm{d}y = 1 \Rightarrow c = 4/63$$

- Determine the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

  For $3 \leq x \leq 4$:

$$f_X(x) = \int_4^5 \frac{4}{63}\,x\,y\,\mathrm{d}y = \frac{4x}{63} \cdot \frac{9}{2} = \frac{2x}{7}$$

  For $4 \leq y \leq 5$:

$$f_Y(y) = \int_3^4 \frac{4}{63}\,x\,y\,\mathrm{d}x = \frac{4y}{63} \cdot \frac{7}{2} = \frac{2y}{9}$$

- Compute the probability that the aircraft flies at an altitude greater than 35000 feet and at an airspeed less than 450 knots.

$$P(X > 3.5,\ Y < 4.5) = \int_{3.5}^4 \int_4^{4.5} \frac{4}{63}\,x\,y\,\mathrm{d}y\,\mathrm{d}x.$$

  The integral factorises:

$$\int_{3.5}^4 x\,\mathrm{d}x = \left[\frac{x^2}{2}\right]_{3.5}^4 = \frac{16 - 12.25}{2} = \frac{15}{8}, \qquad \int_4^{4.5} y\,\mathrm{d}y = \left[\frac{y^2}{2}\right]_4^{4.5} = \frac{20.25 - 16}{2} = \frac{17}{8}.$$

  Therefore,

$$P(X > 3.5,\ Y < 4.5) = \frac{4}{63} \cdot \frac{15}{8} \cdot \frac{17}{8} = \frac{1020}{4032} = \frac{85}{336} \approx 0.253.$$

- What is the expected altitude $\mathrm{E}[X]$ and the expected airspeed $\mathrm{E}[Y]$.

  Using $f_X(x) = \dfrac{2x}{7}$ on $[3,4]$:

$$E[X] = \int_3^4 x \cdot \frac{2x}{7}\,\mathrm{d}x = \frac{2}{7} \cdot \frac{x^3}{3}\bigg|_3^4 = \frac{2}{7} \cdot \frac{64 - 27}{3} = \frac{74}{21} \approx 3.52.$$

Using $f_Y(y) = \frac{2y}{9}$ on $[4, 5]$:

$$E[Y] = \int_4^5 y \cdot \frac{2y}{9}\,\mathrm{d}y = \frac{2}{9} \cdot \frac{y^3}{3}\bigg|_4^5 = \frac{2}{9} \cdot \frac{125-64}{3} = \frac{122}{27} \approx 4.52.$$

- Compute $\mathrm{Cov}(X, Y)$. What does the result confirm about the relationship between X and Y?

  Since $f(x, y) = f_X(x) \cdot f_Y(y)$, the random variables X and Y are **independent**, which immediately implies

  $$\mathrm{Cov}(X, Y) = E[XY] - E[X]\,E[Y] = 0.$$

  This confirms that, within this cruise model, altitude and airspeed are statistically independent: knowledge of the aircraft's altitude provides no information about its airspeed.

6 Cruise Flight

Consider two random variables X and Y that represent the altitude (in kilometers) and airspeed (in Mach) of an aircraft, respectively. The joint probability density function is given by:

$$f(x, y) = \begin{cases} 2, & \text{if } (x, y) \in D, \\ 0, & \text{if } (x, y) \notin D, \end{cases}$$

where $D = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le x\}$.

- Calculate:

  - The marginal distributions of the altitude (X) and airspeed (Y).

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y)\ dy \\ &= \int_0^x 2\ dy \\ &= [2y]_0^x \\ &= 2x \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x, y)\ dx \\ &= \int_y^1 2\ dx \\ &= [2x]_y^1 \\ &= 2 - 2y \end{aligned}$$

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(x)\ dx \\ &= \int_0^x 2x\ dx \\ &= \left[x^2\right]_0^x \\ &= x^2 \end{aligned}$$

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} f_Y(y)\ dy \\ &= \int_0^y 2 - 2y\ dy \\ &= \left[2y - y^2\right]_0^y \\ &= 2y - y^2 \end{aligned}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 2y - y^2 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

- The expected values of the altitude (X) and airspeed (Y).

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 2x^2\, dx = \left[\frac{2x^3}{3}\right]_0^1 = 2/3$$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^1 2y - 2y^2\, dy = \left[\frac{2y^2}{2} - \frac{2y^3}{3}\right]_0^1 = 1/3$$

- The conditional distributions of the altitude (X), given airspeed $Y = y$, and of the airspeed (Y), given altitude $X = x$.

$$f_{X|Y=y}(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{1-y} \quad (y \le x \le 1)$$

$$f_{Y|X=x}(y \mid x) = \frac{f(y,x)}{f_X(x)} = \frac{1}{x} \quad (0 \le y \le x)$$

$$F_{X|Y=y}(x \mid y) = \int_y^x \frac{1}{1-y} dx = \frac{x-y}{1-y} \quad (y \le x \le 1)$$

$$F_{Y|X=x}(y \mid x) = \int_0^y \frac{1}{x} dy = \frac{y}{x} \quad (0 \le y \le x)$$

- The covariance between the altitude (X) and airspeed (Y).

$$\text{Cov}(X,Y) = \iint_{\mathbb{R}^2} (x - \mu_X)(y - \mu_Y) f(x,y)\, dxdy = 2\int_0^1 \int_0^x \left(x - \frac{2}{3}\right)\left(y - \frac{1}{3}\right)\, dydx \approx 0.0278$$

7 Drone Operations

Consider the joint probability density function of two random variables X and Y, where X represents the time (in hours) a drone spends flying, and Y represents the fuel consumption (in liters) during the flight:

$$f(x,y) = \begin{cases} 6(1-x-y), & \text{if } 0 \le y \le 1-x \text{ and } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

- Are the variables X (flight time) and Y (fuel consumption) independent? Justify your answer.
- Compute the cumulative distribution function (CDF) of the random variable X.
- Determine the conditional density function $f_{X|Y=y}(x)$.
- Calculate $P(X < 1/4 \mid Y = 1/2)$, i.e., the probability that the drone's flight time is less than $1/4$ hours given that its fuel consumption is $1/2$ liters.
- Calculate $P(X < 3/4 \mid Y > 1/2)$, i.e., the probability that the drone's flight time is less than $3/4$ hours given that its fuel consumption exceeds $1/2$ liters.

8 Wing Design

The wingspan W and the chord length c of an aircraft are random variables with the following parameters:

$$W : \mu_W = 36\,\text{m},\ \sigma_W = 0.5\,\text{m}$$
$$L : \mu_c = 4\,\text{m},\ \sigma_c = 0.1\,\text{m}$$

- Assuming W and c are independent and that the wing is rectangular, estimate the expected value and the standard deviation for the aircraft's wing area.

9 Manufacturing Process

In an aircraft manufacturing process, oscillation systems are produced for a specific type of aircraft. These systems include a damping mechanism with a resistance R and a capacitor with capacitance C. The oscillation period T of the system depends on R and C according to the following relationship:

$$T = \frac{R^3C^3}{100}.$$

Assume that R and C are random variables with the following parameters:

$$R: \quad \mu_R = 10^6, \quad \sigma_R = 0.03\mu_R,$$
$$C: \quad \mu_C = 10^{-6}, \quad \sigma_C = 0.05\mu_C.$$

Suppose that the resistance and the capacitor incorporated into each oscillation system are selected independently of one another.

- Estimate the expected value and the standard deviation of the oscillation period for these systems.

$$\begin{aligned} T(R,C) \approx\ & T(\mu_R,\mu_C) + \frac{\partial T}{\partial R}(\mu_R,\mu_C)(R-\mu_R) + \frac{\partial T}{\partial C}(\mu_R,\mu_C)(C-\mu_C) \\ &= \frac{(10^6 10^{-6})^3}{100} + \frac{3\mu_R^3\mu_C^2}{100}(R-10^6) + \frac{3\mu_C^2\mu_R^3}{100}(C-10^{-6}) \\ &= -\frac{1}{20} + 3\times 10^{-8}R + 3\times 10^4 C \end{aligned}$$

$$E[T] \approx E\left[-\frac{1}{20} + 3\times 10^{-8}R + 3\times 10^4 C\right] = \frac{1}{100}$$

$$\text{Var}(Z) = \text{Var}(X) + \text{Var}(Y) + \text{Cov}(X,Y)$$

$$\begin{aligned} \text{Var}(T) &\approx \text{Var}\left(-\frac{1}{20} + 3\times 10^{-8}R + 3\times 10^4 C\right) \\ &= \left(3\times 10^{-8}\right)^2 \text{Var}(R) + \left(3\times 10^4\right)^2 \text{Var}(C) \\ &\approx 3.06\times 10^{-6}\ \text{s}^2 \end{aligned}$$