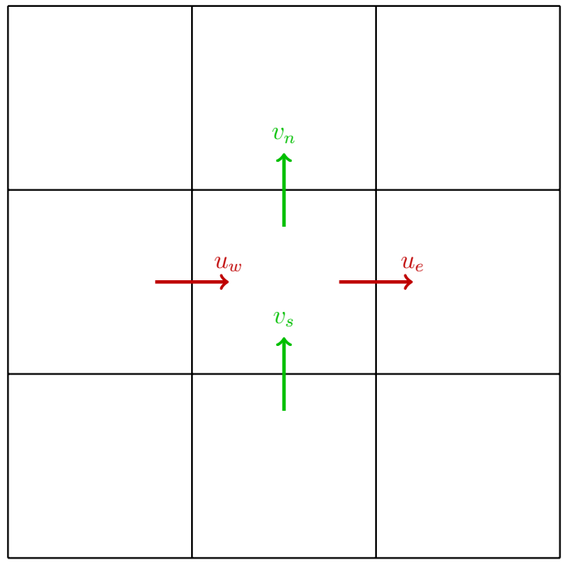
Digesting PDEs!

Komasho

Mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{1}$$

Let us focus on a generic P cell



Let us consider the 2D case ($\partial/\partial z$ are dead!)

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 \tag{2}$$

Assuming no temporal ρ variation

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 \tag{3}$$

Using the Finite Volume Method, let us integrate the equation on the P cell

$$\int_V \nabla \cdot (\rho \vec{u}) \ \mathrm{d}V = \int_V 0 \ \mathrm{d}V \tag{4}$$

$$\int_V \nabla \cdot (\rho \vec{u}) \ \mathrm{d}V = 0 \tag{5}$$

Which using the divergence theorem becomes...

$$\oint_S \rho \vec{u} \cdot \vec{n} \ \mathrm{d}S = 0 \tag{6}$$

Integrating...

$$\rho_e \begin{bmatrix} u_e \\ v_e \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} A_e + \rho_w \begin{bmatrix} u_w \\ v_w \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} A_w + \rho_n \begin{bmatrix} u_n \\ v_n \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} A_n + \rho_s \begin{bmatrix} u_s \\ v_s \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} A_s = 0 \tag{7}$$

which is simplified to...

$$\rho_e u_e A_e - \rho_w u_w A_w + \rho_n v_n A_n - \rho_s v_s A_s = 0 \tag{8}$$

Simplify by introducing the F coefficients

$$F_e - F_w + F_n - F_s = 0 \tag{9}$$

We typically stop here. But why stop here? Using centered differencing...

$$\rho_e \frac{u_P + u_E}{2} A_e - \rho_w \frac{u_P + u_W}{2} A_w + \rho_n \frac{v_P + v_N}{2} A_n - \rho_s \frac{v_P + v_S}{2} A_s = 0 \tag{10}$$

Simplifying...

$$\rho_e \frac{u_P}{2} A_e + \rho_e \frac{u_E}{2} A_e - \rho_w \frac{u_P}{2} A_w - \rho_w \frac{u_W}{2} A_w + \rho_n \frac{v_P}{2} A_n + \rho_n \frac{v_N}{2} A_n - \rho_s \frac{v_P}{2} A_s - \rho_s \frac{v_S}{2} A_s = 0 \tag{11}$$

Organize terms by variable

$$\left[\frac{\rho_e A_e}{2} - \frac{\rho_w A_w}{2} \right] u_P + \left[\frac{\rho_e A_e}{2} \right] u_E - \left[\frac{\rho_w A_w}{2} \right] u_W + \left[\frac{\rho_n A_n}{2} - \frac{\rho_s A_s}{2} \right] v_P + \left[\frac{\rho_n A_n}{2} \right] v_N - \left[\frac{\rho_s A_s}{2} \right] v_S = 0 \tag{12}$$

Congratulations, you transformed a PDE into an Algebraic Equation!

$$a_P^x u_P + a_E^x u_E - a_W^x u_W + a_P^y v_P + a_N^y v_N - a_S^y v_S = 0 \tag{13}$$

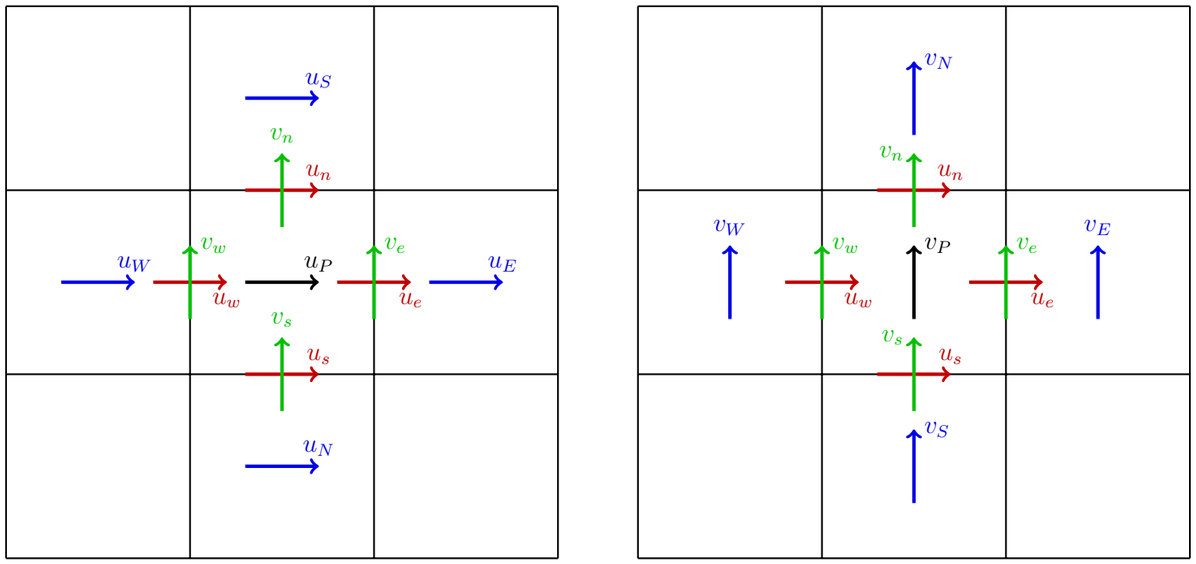
X-Component of Momentum (Repeat for Y-component)

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho f_x \tag{14}$$

Let us consider the 2D case ($\partial/\partial z$ are dead!)

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho uu)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\rho f_x \tag{15}$$

Let us focus on a generic P cell



Using the Finite Volume Method, let us integrate the equation on the P cell

$$\int_V \frac{\partial(\rho u)}{\partial t}+\nabla\cdot(\rho u\vec{u})\,\mathrm{d}V=\int_V -\frac{\partial p}{\partial x}+\mu\nabla\cdot(\nabla u)+\rho f_x\,\mathrm{d}V \tag{16}$$

Rewriting the pressure term as a divergence

$$\int_V \frac{\partial(\rho u)}{\partial t}\,\mathrm{d}V+\int_V \nabla\cdot(\rho u\vec{u})\,\mathrm{d}V=-\int_V \nabla\cdot\left(\begin{bmatrix}p\\0\end{bmatrix}\right)\,\mathrm{d}V+\int_V \mu\nabla\cdot(\nabla u)\,\mathrm{d}V+\int_V \rho f_x\,\mathrm{d}V \tag{17}$$

Let us revisit some useful theorems

$$\boxed{\iiint_V(\nabla\cdot\mathbf{F})\,\mathrm{d}V=\oiint_S(\mathbf{F}\cdot\hat{\mathbf{n}})\,\mathrm{d}S}\ \text{and}\ \boxed{\iiint_V\frac{\partial\phi}{\partial t}\,dV=\frac{d}{dt}\iiint_V\phi\,dV} \tag{18}$$

$$\frac{d}{dt}\int_V \rho u\,\mathrm{d}V+\oint_S \rho u\vec{u}\cdot\vec{n}\,\mathrm{d}S=-\oint_S\begin{bmatrix}p\\0\end{bmatrix}\cdot\vec{n}\,\mathrm{d}S+\oint_S\mu\nabla u\cdot\vec{n}\,\mathrm{d}S+\int_V\rho f_x\,\mathrm{d}V \tag{19}$$

Before moving forward, remember that

$$\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix} \text{ and } \nabla u = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} \tag{20}$$

Integrating...

$$\frac{d}{dt}(\rho_P u_P V_P) + [(\rho u u A)_e - (\rho u u A)_w + (\rho v u A)_n - (\rho v u A)_s] = -(p_e A_e - p_w A_w) + \left[\mu A \frac{\partial u}{\partial x}\right]_e - \left[\mu A \frac{\partial u}{\partial x}\right]_w + \left[\mu A \frac{\partial u}{\partial y}\right]_n - \left[\mu A \frac{\partial u}{\partial y}\right]_s + S_u V_P \tag{21}$$

Approximate viscous terms by centered finite differences

$$\frac{d}{dt}(\rho_P u_P V_P) + (\rho u u A)_e - (\rho u u A)_w + (\rho v u A)_n - (\rho v u A)_s = -(p_e A_e - p_w A_w) + \mu_e A_e \left(\frac{u_E - u_P}{\delta x_{PE}}\right) - \mu_w A_w \left(\frac{u_P - u_W}{\delta x_{WP}}\right) + \mu_n A_n \left(\frac{u_N - u_P}{\delta y_{PN}}\right) - \mu_s A_s \left(\frac{u_P - u_S}{\delta y_{SP}}\right) + S_u V_P \tag{22}$$

Approximate the temporal term with a foward finite difference. Implicit formulation is used.

$$\rho_P V_P \left(\frac{u_P^{t+1} - u_P^t}{\Delta t}\right) + (\rho u u A)_e^{t+1} - (\rho u u A)_w^{t+1} + (\rho v u A)_n^{t+1} - (\rho v u A)_s^{t+1} = -\left(p_e^{t+1} A_e - p_w^{t+1} A_w\right) + \mu_e A_e \left(\frac{u_E^{t+1} - u_P^{t+1}}{\delta x_{PE}}\right) - \mu_w A_w \left(\frac{u_P^{t+1} - u_W^{t+1}}{\delta x_{WP}}\right) + \mu_n A_n \left(\frac{u_N^{t+1} - u_P^{t+1}}{\delta y_{PN}}\right) - \mu_s A_s \left(\frac{u_P^{t+1} - u_S^{t+1}}{\delta y_{SP}}\right) + S_u^{t+1} V_P \tag{23}$$

Push the velocity u out from the convective terms

$$\rho_P V_P \left(\frac{u_P^{t+1} - u_P^t}{\Delta t}\right) + (\rho u A)_e u_e^{t+1} - (\rho u A)_w u_w^{t+1} + (\rho v A)_n u_n^{t+1} - (\rho v A)_s u_s^{t+1} = -\left(p_e^{t+1} A_e - p_w^{t+1} A_w\right) + \mu_e A_e \left(\frac{u_E^{t+1} - u_P^{t+1}}{\delta x_{PE}}\right) - \mu_w A_w \left(\frac{u_P^{t+1} - u_W^{t+1}}{\delta x_{WP}}\right) + \mu_n A_n \left(\frac{u_N^{t+1} - u_P^{t+1}}{\delta y_{PN}}\right) - \mu_s A_s \left(\frac{u_P^{t+1} - u_S^{t+1}}{\delta y_{SP}}\right) + S_u^{t+1} V_P \tag{24}$$

Simplify by introducing the F an D coefficients

$$\rho_P V_P \left(\frac{u_P^{t+1} - u_P^t}{\Delta t}\right) + F_e^{t+1} u_e^{t+1} - F_w^{t+1} u_w^{t+1} + F_n^{t+1} u_n^{t+1} - F_s^{t+1} u_s^{t+1} = -\left(p_e^{t+1} A_e - p_w^{t+1} A_w\right) + D_e^{t+1}\left(u_E^{t+1} - u_P^{t+1}\right) - D_w^{t+1}\left(u_P^{t+1} - u_W^{t+1}\right) + D_n^{t+1}\left(u_N^{t+1} - u_P^{t+1}\right) - D_s^{t+1}\left(u_P^{t+1} - u_S^{t+1}\right) + S_u^{t+1} V_P \tag{25}$$

We need values at the faces. Using centered differencing now.

$$\rho_P V_P \left(\frac{u_P^{t+1} - u_P^t}{\Delta t}\right) + F_e^{t+1}\left(\frac{u_P^{t+1} + u_E^{t+1}}{2}\right) - F_w^{t+1}\left(\frac{u_W^{t+1} + u_P^{t+1}}{2}\right) + F_n^{t+1}\left(\frac{u_P^{t+1} + u_N^{t+1}}{2}\right) - F_s^{t+1}\left(\frac{u_S^{t+1} + u_P^{t+1}}{2}\right) =$$
$$-\left(p_e^{t+1} A_e - p_w^{t+1} A_w\right) + D_e^{t+1}\left(u_E^{t+1} - u_P^{t+1}\right) - D_w^{t+1}\left(u_P^{t+1} - u_W^{t+1}\right) + D_n^{t+1}\left(u_N^{t+1} - u_P^{t+1}\right) - D_s^{t+1}\left(u_P^{t+1} - u_S^{t+1}\right) + S_u^{t+1} V_P \tag{26}$$

Simplify!

$$\frac{\rho_P V_P}{\Delta t} u_P^{t+1} - \frac{\rho_P V_P}{\Delta t} u_P^t + \frac{F_e^{t+1}}{2} u_P^{t+1} + \frac{F_e^{t+1}}{2} u_E^{t+1} - \frac{F_w^{t+1}}{2} u_W^{t+1} - \frac{F_w^{t+1}}{2} u_P^{t+1} + \frac{F_n^{t+1}}{2} u_P^{t+1} + \frac{F_n^{t+1}}{2} u_N^{t+1} - \frac{F_s^{t+1}}{2} u_S^{t+1} - \frac{F_s^{t+1}}{2} u_P^{t+1} =$$
$$-\left(p_e^{t+1} A_e - p_w^{t+1} A_w\right) + D_e^{t+1} u_E^{t+1} - D_e^{t+1} u_P^{t+1} - D_w^{t+1} u_P^{t+1} + D_w^{t+1} u_W^{t+1} + D_n^{t+1} u_N^{t+1} - D_n^{t+1} u_P^{t+1} - D_s^{t+1} u_P^{t+1} + D_s^{t+1} u_S^{t+1} + S_u^{t+1} V_P \tag{27}$$

Organize terms by variable

$$\left[\frac{\rho_P V_P}{\Delta t}+\frac{F_e^{t+1}}{2}-\frac{F_w^{t+1}}{2}+\frac{F_n^{t+1}}{2}-\frac{F_s^{t+1}}{2}+D_e^{t+1}+D_w^{t+1}+D_n^{t+1}+D_s^{t+1}\right]u_P^{t+1}+\left[\frac{F_e^{t+1}}{2}-D_e^{t+1}\right]u_E^{t+1}+\left[-\frac{F_w^{t+1}}{2}-D_w^{t+1}\right]u_W^{t+1}+\left[\frac{F_n^{t+1}}{2}-D_n^{t+1}\right]u_N^{t+1}+\left[-\frac{F_s^{t+1}}{2}-D_s^{t+1}\right]u_S^{t+1}$$
$$=-\left(p_e^{t+1}A_e-p_w^{t+1}A_w\right)+S_u^{t+1}V_P+\frac{\rho_P V_P}{\Delta t}u_P^t \quad (28)$$

Congratulations, you transformed a PDE into an Algebraic Equation!

$$\boxed{a_P u_P^{t+1}+a_E u_E^{t+1}+a_W u_W^{t+1}+a_N u_N^{t+1}+a_S u_S^{t+1}=b} \quad (29)$$

with

$$\begin{aligned}
a_P &= \frac{\rho_P V_P}{\Delta t}+\frac{F_e^{t+1}-F_w^{t+1}+F_n^{t+1}-F_s^{t+1}}{2}+D_e^{t+1}+D_w^{t+1}+D_n^{t+1}+D_s^{t+1}\\
a_E &= -D_e^{t+1}+\frac{F_e^{t+1}}{2}\\
a_W &= -D_w^{t+1}-\frac{F_w^{t+1}}{2}\\
a_N &= -D_n^{t+1}+\frac{F_n^{t+1}}{2}\\
a_S &= -D_s^{t+1}-\frac{F_s^{t+1}}{2}\\
b &= \frac{\rho_P V_P}{\Delta t}u_P^t-\left(p_e^{t+1}A_e-p_w^{t+1}A_w\right)+S_u^{t+1}V_P
\end{aligned} \quad (30)$$

Repeat for the Y-Component of Momentum! And Z-component if you dare!

Visual Example

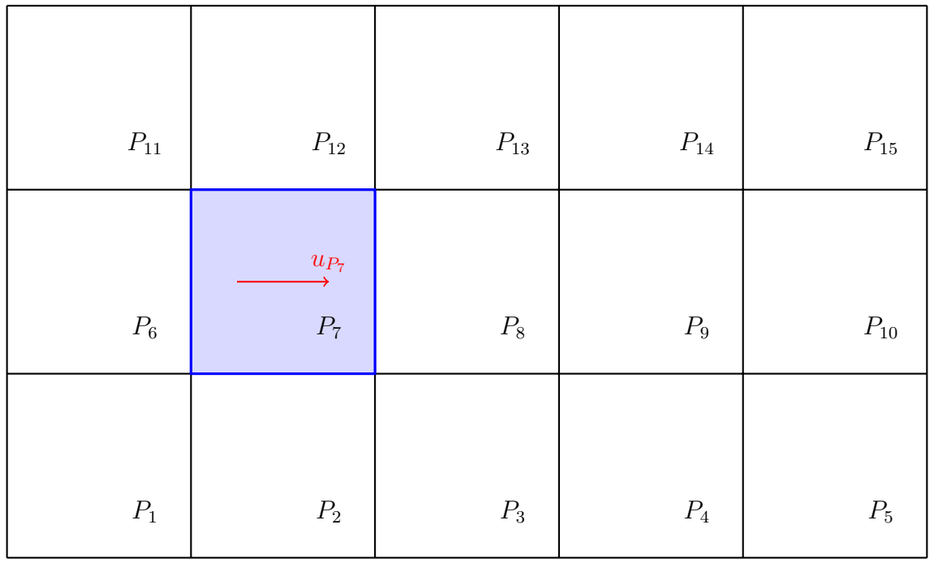


Figure 1: Five by three structured finite-volume mesh.

$$
\begin{bmatrix}
a_{P_1} & -a_{E_1} & 0 & 0 & 0 & -a_{N_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{W_2} & a_{P_2} & -a_{E_2} & 0 & 0 & 0 & -a_{N_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -a_{W_3} & a_{P_3} & -a_{E_3} & 0 & 0 & 0 & -a_{N_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -a_{W_4} & a_{P_4} & -a_{E_4} & 0 & 0 & 0 & -a_{N_4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a_{W_5} & a_{P_5} & 0 & 0 & 0 & 0 & -a_{N_5} & 0 & 0 & 0 & 0 & 0 \\
-a_{S_6} & 0 & 0 & 0 & 0 & a_{P_6} & -a_{E_6} & 0 & 0 & 0 & -a_{N_6} & 0 & 0 & 0 & 0 \\
0 & -a_{S_7} & 0 & 0 & 0 & -a_{W_7} & a_{P_7} & -a_{E_7} & 0 & 0 & 0 & -a_{N_7} & 0 & 0 & 0 \\
0 & 0 & -a_{S_8} & 0 & 0 & 0 & -a_{W_8} & a_{P_8} & -a_{E_8} & 0 & 0 & 0 & -a_{N_8} & 0 & 0 \\
0 & 0 & 0 & -a_{S_9} & 0 & 0 & 0 & -a_{W_9} & a_{P_9} & -a_{E_9} & 0 & 0 & 0 & -a_{N_9} & 0 \\
0 & 0 & 0 & 0 & -a_{S_{10}} & 0 & 0 & 0 & -a_{W_{10}} & a_{P_{10}} & 0 & 0 & 0 & 0 & -a_{N_{10}} \\
0 & 0 & 0 & 0 & 0 & -a_{S_{11}} & 0 & 0 & 0 & 0 & a_{P_{11}} & -a_{E_{11}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_{S_{12}} & 0 & 0 & 0 & -a_{W_{12}} & a_{P_{12}} & -a_{E_{12}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{S_{13}} & 0 & 0 & 0 & -a_{W_{13}} & a_{P_{13}} & -a_{E_{13}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{S_{14}} & 0 & 0 & 0 & -a_{W_{14}} & a_{P_{14}} & -a_{E_{14}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{S_{15}} & 0 & 0 & 0 & -a_{W_{15}} & a_{P_{15}}
\end{bmatrix}
\begin{bmatrix}
u_1^{t+1} \\ u_2^{t+1} \\ u_3^{t+1} \\ u_4^{t+1} \\ u_5^{t+1} \\ u_6^{t+1} \\ u_7^{t+1} \\ u_8^{t+1} \\ u_9^{t+1} \\ u_{10}^{t+1} \\ u_{11}^{t+1} \\ u_{12}^{t+1} \\ u_{13}^{t+1} \\ u_{14}^{t+1} \\ u_{15}^{t+1}
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \\ b_9 \\ b_{10} \\ b_{11} \\ b_{12} \\ b_{13} \\ b_{14} \\ b_{15}
\end{bmatrix}
\tag{31}
$$